Solution of MRI Diffusion Equation via the Adomian Decomposition Method

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Abstract

In this paper two nonlinear equation that they are about diffusion and Magnetic Resonance Imaging (MRI) is defined and solution of them is obtained via Adomian Decomposition Method (ADM). Finally solutions are in good agreement with exact solutions.

Keywords: MRI, ADM, Exact solution, Medical

1. Introduction

Magnetic resonance imaging (MRI) is a medical imaging technique used in radiology to form pictures of the anatomy and the physiological processes of the body in both health and disease. MRI scanners use strong magnetic fields, radio waves, and field gradients to generate images of the inside of the body. MRI is based upon the science of nuclear magnetic resonance (NMR). Certain atomic nuclei can absorb and emit radio frequency energy when placed in an external magnetic field. In clinical and research MRI, hydrogen atoms are most-often used to generate a detectable radio-frequency signal that is received by antennas in close proximity to the anatomy being examined. Hydrogen atoms exist naturally in people and other biological organisms in abundance, particularly in water and fat. For this reason, most MRI scans essentially map the location of water and fat in the body. Pulses of radio waves excite the nuclear spin energy transition, and magnetic field gradients localize the signal in space[3]. By varying the parameters of the pulse sequence, different contrasts can be generated between tissues based on the relaxation properties of the hydrogen atoms therein. Now days MRI is improved a lot. The success of diffusion MRI is deeply rooted in the powerful concept that during their random, diffusion driven displacements molecules probe tissue structure at a microscopic scale well beyond the usual image resolution. The diffusion MRI can reveal orientation-dependent behavior of water molecules for localization of specific organs and pathologies, and for functionality assessment.[4] As diffusion is truly a three dimensional process, molecular mobility in tissues may be anisotropic, as in brain white matter. With diffusion tensor imaging (DTI), diffusion anisotropy effects can be fully extracted, characterized, and exploited, providing even more exquisite details on tissue microstructure. On the other hand, measuring diffusion in the real space, for instance water diffusion in a living body, is not a simple task though it provides useful and important information. Indeed, incoherent motion of water molecules has certain anisotropy in living bodies relating to normal and abnormal structures.[6] The diffusion may be one of the most important and attractive notions in mathematical
methods for image analysis. Diffusion equation is one of the most important models which appears in the MRI, and often is nonlinear. Nonlinear partial differential

Most nonlinear models of real life problems are still very difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to the nonlinear models.

Now in this paper ADM is selected because is an effective procedure for a semi-analytical solution of a wide range of dynamical systems. It is based on decompositions of the operator and the solution, and does not require linearization, or weak nonlinearity assumptions.

2. Basic method

First the main form of equation introduce as,

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u + au + N(u) = f(x,t)$$

Where "a" is trial cte, "N" is nonlinear terms and u=u(x,t)

$$u(x,0) = b_0(x) \text{and } \frac{\partial u(x,t)}{\partial t} \bigg|_{t=0} = b_1(x)$$

Now

$$u = \sum_{n=0}^{\infty} u_n, N(u) = \sum_{n=0}^{\infty} A_n(u_0,u_1,...,u_n)$$

And \( A_n \) is,

$$A_n = \frac{1}{n!} \frac{d^n}{dh^n} \left[ N \left( \sum_{r=0}^{\infty} h^r u_r \right) \right] \bigg|_{h=0} = \sum_{r=1}^{\infty} C (r,n) N^r (u_0), n = 0,1,2,$$

With

$$A_0 = g(u_0), A_1 = u_1 \frac{d}{du_0} g(u_0), A_2 = u_2 \frac{d}{du_0} g(u_0) + \frac{u_1^2}{2} \frac{d^2}{du_0^2} g(u_0).$$

we define the linear operators

$$L_0 = \frac{\partial^2}{\partial t^2}, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Eq(1) can be write as

$$L_0 u = \nabla^2 u - au - N(u) + f$$

Again if we define the inverse operator \( L_i^{-1} \), above eq convert to,

$$L_i^{-1} L_0 u = L_i^{-1} (\nabla^2 u) - aL_i^{-1} u - L_i^{-1} [N(u)] + L_i^{-1} f$$

Now

$$u(x,t) = b_0(x) + b_1(x) + L_0^{-1} (\nabla^2 u) - aL_0^{-1} u - L_0^{-1} [N(u)] + L_0^{-1} f$$
Where Lt-1 is defined as,

\[ L_{t-1}^{-1} = \int_0^t \int_0^t dt' dt \]

Term \( u_0 \) is as,

\[ u_0 = b_0(x) + b_1(x)t + L_{t-1}^{-1} \{ f(x,t) \} \]

So others “u” can be determined as,

\[ u_{n+1} = L_{t-1}^{-1}(\nabla^2 u_n) - aL_{t-1}^{-1}(u_n) - L_{t-1}^{-1}\{ N(u_n) \}, n \geq 0 \] (11)

The expression

\[ \phi_n = \sum_{i=0}^{n-1} u_i(x,t) \] (12)

The convergence is very rapidly. [2,7,1,8]

3. Problems

First problem is

\[ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2\pi u + u^5 = x \sin^2 \left( \frac{\pi t}{2} \right) \] (13)

\[ u(x,0) = 0, u_x(x,0) = 2\pi x \] (14)

After computation,

\[ u_0 = \frac{\pi xt}{2} + x^2 \left( \frac{\pi^4 t^4}{48} - \frac{\pi^4 t^6}{1271} \right) + x^3 \left( \frac{\pi t^3}{160} - \frac{\pi^3 t^5}{2348} \right) \]

\[ u_1 = -0.216\pi xt^3 - \frac{x^2 \pi^2 t^4}{27} - \frac{x^4 \pi^4 t^5}{121} \]

\[ + \frac{1}{220}(0.001276x^6 \pi^6 - 0.0013634 \pi^6 x^4) \]

\[ u_2 = -0.0012(0.75(0.00569x \pi^4 - 0.0053432x^2 \pi^3))x^2 \pi^2 \]

\[ + (-0.0248x^2 - 0.0456x^4 \pi^4) + 0.006759x \pi^4) \pi x + \ldots \]

The solution is,

\[ \phi_n = \sum_{i=0}^{n-1} u_i(x,t) \]

Table 1 compare the ADM with analytical.
**Table 1:** comparing ADM (D) with exact (E)

<table>
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<th>X=0.2</th>
<th>X=0.3</th>
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<td>0.0312435E</td>
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<td>0.1175565D</td>
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</table>

**Second problem is as below,**

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 3u + 5u^2 = 8e^{-9t} + x^2 e^{-9t}
\]

\[u(x,0) = x^2, u_t(x,0) = -9x^2\]

After computation,

\[u_0 = x^2 - 9x^4t - t^2 + \frac{9t^4}{3} - \frac{t^5}{12} + \frac{9t^6}{22} - \frac{3t^7}{13}\]

\[\frac{3t^8}{143} - \frac{\sqrt{6t^9}}{840}\]

\[u_1 = -9x^3t + \frac{9x^4t^2}{12} - \frac{3t^4}{4} + \frac{3t^5}{11} + \frac{\sqrt{6t^7}}{70} - \frac{t^4}{12} - \frac{t^5}{10}\]

\[-\frac{8t^8}{281} + \frac{\sqrt{6t^9}}{840} - \frac{t^{10}}{1120} + \ldots\]

\[u_2 = 1.262 \times 10^{-4} t^{28} - 1.127 \times 10^{-11} t^{23} + 5.547 \times 10^{-8} t^{25} - 5.567 \times 10^{-9} t^{32} - 2.824 \times 10^{-7} t^{30} - 2.774 \times 10^{-10} \sqrt{6t^{26}} + 2.427 \times 10^{-10} 9t^{23} - 5.313 \times 10^{-9} \sqrt{6t^{22}} + \ldots\]

After summation of different “u”, the solution and answer is achieved.

\[\phi_n = \sum_{i=0}^{n-1} u_i(x, t)\]

**Table 2** compare the ADM with analytical.
Conclusion
In this paper we applied the ADM to solving the diffusion equation that have application is MRI and the result is that it works very good and do not need any linearization, simplification and other operations. So this method can derive in MRI device for making a better image from body.

References


