Analytical Study on Nonlinear Vibration of Carbon Nanotubes Using Nonlocal Timoshenko Beam Theory

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Abstract

In this paper, variational iteration (VIM) and energy balance (EBM) methods have been used to investigate non-linear vibration of Timoshenko beam theory. The proposed methods do not require small parameter in the equation which is difficult to be found for nonlinear problems. Comparison of VIM and EBM with Runge-Kutta 4th leads to highly accurate solutions.

Keywords: Variational Iteration Method (VIM), Energy balance Method (EBM), Galerkin method, non-linear vibration, Timoshenko beam theory.

1. Introduction

The demand for engineering structures is continuously increasing. Aerospace vehicles, bridges, and automobiles are examples of these structures. Many aspects have to be taken into consideration in the design of these structures to improve their performance and extend their life. One aspect of the design process is the dynamic response of structures. The dynamics of distributed-parameter and continuous systems, like beams, were governed by linear and nonlinear partial-differential equations in space and time. It was difficult to find the exact or closed-form solutions for nonlinear problems. Consequently, researchers were used two classes of approximate solutions of initial boundary-value problems: numerical techniques [28, 31], and approximate analytical methods [2, 26]. For strongly non-linear partial-differential, direct techniques, such as perturbation methods, were not utilized to solve directly the non-linear partial-differential equations and associated boundary conditions. Therefore first partial-differential equations are discretized into a set of non-linear ordinary-differential equations using the Galerkin approach and the governing problems are then solved analytically in time domain. Approximate methods for studying non-linear vibrations of beams are important for investigating and designing purposes. In recent years, some promising approximate analytical solutions have been proposed, such as Frequency Amplitude Formulation [13], Variational Iteration [5, 6, 14, 17], Homotopy-Perturbation [3, 4, 7, 24], Parametrized-Perturbation [18], Max-Min [15, 19, 29], Differential Transform Method [16], Adomian Decomposition Method [22], Energy Balance [23, 30], etc.
2. Nonlinear vibration analysis of nonlocal SWCNTs

Fig. 1 shows a SWCNT modeled as a Timoshenko nanobeam with length \( L \), radius \( r \), and effective tube thickness \( h \). It is assumed that the SWCNTs vibrate only in the \( x-z \) plane. Based on Timoshenko beam theory, the displacements of an arbitrary point in the beam along the \( x \)- and \( z \)-axes, denoted by \( U(x, z, t) \) and \( W(x, z, t) \), respectively, are \( U(x, t) + z\psi(x, t) \), \( W(x, z, t) = W(x, t) \). Where \( U(x, t) \) and \( W(x, t) \) are displacement components in the midplane, \( \psi \) is the rotation of beam cross-section and \( t \) is time. The nonlinear equations of motion for the nonlocal SWCNTs modeled as a Timoshenko nanobeam is given by:

\[
\begin{align*}
EA \left( \frac{\partial^2 U}{\partial x^2} + \left( \frac{\partial W}{\partial x} \right) \left( \frac{\partial^2 W}{\partial x^2} \right) \right) &= \rho A \frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial t} \left( (e_0a)^2 \right) -\frac{\partial^2}{\partial x^2} \left( (e_0a)^2 \right) \quad (1) \\
K_GA \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial \psi}{\partial x} \right) + \Sigma_1 - \frac{\partial}{\partial t} \left( (e_0a)^2 \right) &= \rho A \frac{\partial^2 W}{\partial t^2} \left( W - (e_0a)^2 \right) \quad (2) \\
\rho A(e_0a)^2 \frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial t} \left( (e_0a)^2 \right) &= \rho A(e_0a)^2 \frac{\partial^2 W}{\partial t^2} \left( W - (e_0a)^2 \right) \quad (3)
\end{align*}
\]

where \( A \) is the cross-sectional area of the beam, \( I \) is the second moment of area and \( \rho \) is the mass density of beam material, \( E \) and \( G \) are Young’s modulus and shear modulus, respectively. The constitutive relations in classical elasticity theories can be recovered by setting the nonlocal parameter \( e_0a = 0 \) and \( K_G \) is the shear correction factor depending on the shape of the cross-section of the beam. Introducing the following dimensionless quantities:

\[
\begin{align*}
\zeta &= \frac{x}{L} \quad (u, w) = \left( \frac{U, W}{r} \right) \quad \psi = \psi \quad (I_1, I_3) = \left( \frac{pA}{\rho A^2}, \frac{\rho l}{\rho Ar^2} \right) \quad \eta = \frac{L}{r} \\
\mu &= \frac{e_0a}{L} \quad (a_{11}, a_{55}, d_{11}) = \left( \frac{EA}{EA}, \frac{K_G A}{EA}, \frac{EI}{EA^2} \right) \quad \tau = \frac{t}{L \sqrt{\rho}}
\end{align*}
\]

(4)
Eq. (1),(2),(3) can be expressed in dimensionless form as:

\[ a_{11} \left( \frac{\partial^2 u}{\partial \xi^2} + \frac{1}{\eta} \frac{\partial w}{\partial \xi} \right) \left( \frac{\partial^2 w}{\partial \xi^2} \right) = I_1 \frac{\partial^2}{\partial \tau^2} \left( u - \mu^2 \left( \frac{\partial^2 u}{\partial \xi^2} \right) \right) \]  

(5)

\[ a_{55} \left( \frac{\partial^2 w}{\partial \xi^2} + \frac{\eta \psi}{\partial \xi} \right) + \Sigma_1 - \mu^2 \Sigma_2 = I_1 \frac{\partial^2}{\partial \tau^2} \left( w - \mu^2 \left( \frac{\partial^2 w}{\partial \xi^2} \right) \right) - I_1 \frac{\partial^2}{\partial \tau^2} \left( \frac{\partial^2 u}{\partial \xi^2} \right) - I_1 \frac{\partial^2}{\partial \tau^2} \left( \frac{\partial^2 u}{\partial x^2} \right) \]  

(6)

\[ d_{11} \frac{\partial^2 \psi}{\partial \xi^2} - a_{55} \eta \left( \frac{\partial w}{\partial \xi} + \eta \psi \right) = I_3 \frac{\partial^2}{\partial \tau^2} \left( \psi - \mu^2 \left( \frac{\partial^2 \psi}{\partial \xi^2} \right) \right) \]  

(7)

Where

\[ \Sigma_1 = \frac{a_{11}}{\eta} \left( \frac{\partial^2 u}{\partial \xi^2} \left( \frac{\partial^2 u}{\partial \xi^2} \right) \right) + \frac{3}{2 \eta} \left( \frac{\partial w}{\partial \xi} \right)^2 \left( \frac{\partial^2 u}{\partial \xi^2} \right) \]  

(8)

\[ \Sigma_2 = \frac{a_{11}}{\eta} \left( \frac{\partial^4 u}{\partial \xi^4} \left( \frac{\partial^2 w}{\partial \xi^2} \right) \right) + 3 \left( \frac{\partial^3 u}{\partial \xi^3} \right) \left( \frac{\partial^2 w}{\partial \xi^2} \right) + 3 \left( \frac{\partial^2 u}{\partial \xi^2} \right) \left( \frac{\partial^3 w}{\partial \xi^3} \right) \]  

(9)

Assuming \( \psi(\xi, \tau) = u(\xi) v(\tau) \) and \( w(\xi, \tau) = f(\xi) g(\tau) \) with ignore \( U \), where \( u(\xi) \), \( f(\xi) \) is the first eigenmode of the beam [32] and applying the Galerkin method, the equation of motion is obtained as follows:

\[ \ddot{g}(\tau) + \alpha_1 g(\tau) + \alpha_2 v(\tau) + \alpha_3 g^3(\tau) = 0 \]  

(8)

\[ \ddot{v}(\tau) + \beta_1 v(\tau) + \beta_2 g(\tau) = 0 \]  

(9)

Where \( \alpha_1, \alpha_2, \alpha_3, \beta_1 \) and \( \beta_2 \) are as follows:

\[ \alpha_1 = -a_{55} \frac{\int_0^1 f(\xi) v(\xi) d\xi}{\int_0^1 \frac{f(\xi)}{d\xi} d\xi}, \quad \alpha_2 = -a_{55} \frac{\int \eta f(\xi) v(\xi) u(\xi) d\xi}{\int \eta f(\xi) u(\xi) d\xi}, \quad \alpha_3 = -\frac{3a_{11} \int_0^1 f(\xi) v(\xi) u(\xi) d\xi}{2 \eta^2 \int_0^1 f(\xi) u(\xi) d\xi}, \]  

(10)

The Eq. (8),(9) is the governing non-linear vibration of Timoshenko beams.

The center of the beam subjected to the following initial conditions:

\[ g(0) = A, \quad g(0) = 0, \quad v(0) = B, \quad \dot{v}(0) = 0 \]  

(11)

where \( A, B \) denotes the non-dimensional maximum amplitude of oscillation.

3. BASIC IDEA OF VARIATIONAL ITERATION METHOD

To illustrate the basic concepts of the VIM, we consider the following differential equation:

\[ Lu + Nu = g(t) \]  

(12)

Where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(t) \) an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\eta) + Nu_n(\eta) - g(\eta))d\eta \]  

(13)
Where $\lambda$ is a general Lagrange multiplier which can be identified optimally via the variational theory [17]. The subscript $n$ indicates the $n$th approximation and $\delta \tilde{u}_n$ is considered as a restricted variation [17], i.e. $\delta \tilde{u}_n = 0$.

4. APPLICATION OF VARIATIONAL ITERATION METHOD

To solve Eq. (8),(9) by means of VIM, we start with an arbitrary initial approximation:

$$g_0(\tau) = A \cos(\omega \tau) , \quad v_0(\tau) = B \cos(\omega \tau)$$  \hspace{1cm} (14)

From Eq. (8), we have:

$$\ddot{g}(\tau) = -\alpha_1 g(\tau) - \alpha_2 v(\tau) - \alpha_3 g^3(\tau) \Rightarrow$$

$$\ddot{g}(\tau) = -\alpha_1 A \cos(\omega \tau) - \alpha_2 B \cos(\omega \tau) - \alpha_3 A^3 \cos^3(\omega \tau)$$  \hspace{1cm} (15)

Integrating twice yields:

$$g_1(\tau) = \frac{\alpha_1 A \cos(\omega \tau)}{\omega^2} + \frac{\alpha_2 B \cos(\omega \tau)}{\omega^2} + \frac{\alpha_3 A^3}{4} \left( \frac{\cos(3\omega \tau)}{9\omega^2} + \frac{3 \cos(\omega \tau)}{\omega^2} \right)$$  \hspace{1cm} (16)

Equating the coefficients of $\cos(\omega \tau)$ in $g_0$ and $g_1$, we have:

$$\omega_{VIM} = \sqrt{\alpha_1 + \frac{\alpha_2 B}{A} + 0.75 \alpha_3 A^2}$$  \hspace{1cm} (17)

And therefore,

$$g_0(\tau) = A \cos\left( \frac{\alpha_1}{\omega} + \frac{\alpha_2 B}{A} + 0.75 \alpha_3 A^2 \tau \right)$$  \hspace{1cm} (18)

Where $\delta \tilde{u}_n = 0$ is considered as restricted variation.

$$g_{n+1}(\tau) = g_n(\tau) + \int_0^\tau \lambda \left( \frac{d^2 g_n}{d\eta^2} + \alpha_1 g_n + \alpha_2 v_n + \alpha_3 g_n^3 \right) d\eta$$  \hspace{1cm} (19)

$$v_{n+1}(\tau) = v_n(\tau) + \int_0^\tau \lambda \left( \frac{d^2 v_n}{d\eta^2} + \beta_1 v_n + \beta_2 g_n \right) d\eta$$  \hspace{1cm} (20)

Its stationary conditions can be obtained as follows:

$$1 - \lambda'\big|_{\eta=\tau} = 0$$  \hspace{1cm} (21)

$$\lambda\big|_{\eta=\tau} = 0$$  \hspace{1cm} (22)

$$\lambda'' + \omega^2 \lambda = 0$$  \hspace{1cm} (23)

Therefore, the multiplier, can be identified as

$$\lambda = \frac{1}{\omega} \sin(\omega (\eta - \tau))$$  \hspace{1cm} (24)

As a result, we obtain the following iteration formula:

$$g_{n+1}(\tau) = g_n(\tau) + \int_0^\tau \frac{1}{\omega} \sin(\omega (\eta - \tau)) \left( \frac{d^2 g_n}{d\eta^2} + \alpha_1 g_n + \alpha_2 v_n + \alpha_3 g_n^3 \right) d\eta$$  \hspace{1cm} (25)

$$v_{n+1}(\tau) = v_n(\tau) + \int_0^\tau \frac{1}{\omega} \sin(\omega (\eta - \tau)) \left( \frac{d^2 v_n}{d\eta^2} + \beta_1 v_n + \beta_2 g_n \right) d\eta$$  \hspace{1cm} (26)

By the iteration formula (25),(26), we can directly obtain other components as:
\[ g_1(\tau) = A \cos(\omega_1 \tau) + \frac{1}{32 \omega^2} (-\alpha_3 A^3 \cos(\omega_1 \tau) + 16 \omega_3 A \tau \sin(\omega_1 \tau) - 16 \omega_2 B \tau \sin(\omega_1 \tau) - 12 \omega_3 A^3 \tau \sin(\omega_1 \tau) + \alpha_3 A^3 \cos(3 \omega_1 \tau) ) \]  

Where \( \omega \) is evaluated from Eq. (17). In the same manner, the rest of the components of the iteration formula can be obtained.

5. APPLICATION OF ENERGY BALANCE METHOD

In order to assess the advantages and the accuracy of the energy balance method; we will apply this method to the discussed system.

Equation of motion, which reads:

\[ g''(\tau) + \alpha_1 g'(\tau) + \alpha_2 v(\tau) + \alpha_3 g^3(\tau) = 0 \]  

(28)

\[ v''(\tau) + \beta_1 v(\tau) + \beta_2 g(\tau) = 0 \]  

(29)

\[ g(0) = E, \; g'(0) = 0, \; v(0) = B, \; v'(0) = 0 \]  

(30)

Its variational formulation can be easily established:

\[ J(u) = \int_0^\tau \left( \frac{g'^2}{2} + \alpha_1 \frac{g^2}{2} + \alpha_2 v g + \alpha_3 \frac{g^4}{4} \right) d\tau \]  

(31)

It’s Hamiltonian, therefore, can be written in the form

\[ H = \frac{g'^2}{2} + \alpha_1 \frac{g^2}{2} + \alpha_2 v g + \alpha_3 \frac{g^4}{4} \]  

(32)

and

\[ H_{\tau=0} = \alpha_1 \frac{A^2}{2} + \alpha_2 AB + \alpha_3 A^4 \frac{4}{4} \]  

(33)

\[ H_{\tau} - H_{\tau=0} = \frac{g'^2}{2} + \alpha_1 \frac{g^2}{2} + \alpha_2 v g + \alpha_3 \frac{g^4}{4} - \alpha_1 \frac{A^2}{2} - \alpha_2 AB - \alpha_3 A^4 \frac{4}{4} \]  

(34)

We will use the trial function to determine the angular frequency \( \omega \), i.e.,

\[ g_0(\tau) = A \cos(\omega \tau) , \; v_0(\tau) = B \cos(\omega \tau) \]  

(35)

If we substitute (35) into (34), it results the following residual equation:

\[ \frac{(A \omega \sin(\omega \tau))^2}{2} + \alpha_1 \frac{(A \omega \sin(\omega \tau))^2}{2} + \alpha_2 AB (\cos(\omega \tau))^2 + \alpha_3 \frac{(A \omega \sin(\omega \tau))^4}{4} - \alpha_1 \frac{A^2}{2} - \alpha_2 AB - \alpha_3 A^4 \frac{4}{4} = 0 \]  

(36)

If we collocate at \( \omega \tau = \frac{\pi}{4} \), we obtain:

\[ \alpha_1 \frac{A^2}{4} + \alpha_2 AB \frac{A}{2} + \alpha_3 A^4 \frac{1}{16} - \alpha_1 \frac{A^2}{2} - \alpha_2 AB - \alpha_3 A^4 \frac{4}{4} = 0 \]  

(37)

or

\[ \omega_{EBM} = \sqrt{\frac{\alpha_1}{A} + \frac{2 \alpha_2 B}{A} + 0.75 \alpha_3 A^2} \]  

(38)

6. RESULTS AND DISCUSSIONS

The behavior of \( g(A, t) \) obtained by VIM and EBM frequency at \( \alpha_1=2.910191164, \; \alpha_2=9.828472866, \; \alpha_3=0.4357628396 \) and B=0,0.005 is shown in Figs.2, 3 and 4. Results of VIM and EBM frequency versus amplitude has been investigated and plotted in Figs.5, 6. Influence of B on
frequency (VIM & EBM) and amplitude has been depicted in Figs.7, 8. The solutions are also compared for \( t=0.5 \) in Table1. It can be observed that there is an excellent agreement between the results obtained from VIM and EBM with those of Runge-Kutta 4th order method [1].

**Figure 2**: VIM & EBM deflection at \( \alpha_1 = 2.910191164 \), \( \alpha_2 = 9.828472866 \), \( \alpha_3 = 0.4357628396 \) for \( B=0 \)

**Figure 3**: VIM deflection at \( \alpha_1 = 2.910191164 \), \( \alpha_2 = 9.828472866 \), \( \alpha_3 = 0.4357628396 \) for \( B=0.005 \)
Figure 4: EBM deflection at $\alpha_1 = 2.910191164$, $\alpha_2 = 9.828472866$, $\alpha_3 = 0.4357628396$ for $B=0.005$

Figure 5: Results of VIM frequency versus amplitude
Figure 6: Results of EBM frequency versus amplitude

Figure 7: Results of VIM frequency versus amplitude associated with influence of B
Table 1: comparison between VIM & EBM with time marching solution for motion equation (8)

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Conclusions

In this work energy balance method (EBM) and variational iteration method (VIM) are proved to be very convenient and powerful mathematical tools to solving nonlinear oscillators and the solutions obtained are in good agreement with numerical values.

References


