Performance Analysis of Microsystems in the Presence of Pull-In Instability

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Abstract

Objective of the current work is to represent performance analysis of microsystems in the presence of pull-in instability phenomenon. To this end, the Euler-Bernoulli beam theory and non-linear Reynolds equation are considered, and the generalized differential quadrature method (GDQM) is employed to solve the nonlinear partial differential governing equations. A parametric study conducted, focusing on the combined effects of geometric non-linearity, Squeeze-film damping (SQFD), distributed load, inertial gap, and material properties. The results reveal that the pull-in instability of poly-silicon micro-beam is strongly size-dependent. It is also demonstrated that neglecting pressure in Reynolds equation will cause smaller values in pull-in time.

Keywords: Micro-beam, Pull-in instability, Reynolds equation, Dimensionless variables.

1. Introduction

Micro electromechanical systems (MEMS) have many applications in the field of engineering and science because of their small size, low unit cost, and high volume production. In recent years, MEMS is gaining great interest among researchers [1, 2]. As ability of MEMS is well reflected in rapidly growing research in various types of systems, the approach to realization of such systems has mostly been processed centric [3]. For instance, micro-scale movable mechanical elements, such as springs, gears, cranks, sliders, and turbines can be produced using IC-compatible micro-fabrication technology [4]. One of the main restrictions on the usage of these systems lies in the pull-in phenomenon, which is a structural instability from the interaction between elastic and electrostatic forces [5]. On the other hand, the pull-in instability limits the range of stable equilibrium states of the oscillatory system and therefore limits the performance of MEMS[6]. The critical value of voltage and displacement corresponding to this instability is referred to as the pull-in voltage and displacement, respectively [7, 8]. It is worth noting that exact nomination of pull-in voltage is essential in the design process to evaluate the sensitivity, frequency response, and instability of such systems [9]. As we know, instability means not more than the loss of stability and post-critical response of a system, and it also consists of static and dynamic instabilities [10]. In this regard, various analytical, experimental, and numerical studies have been conducted on the pull-in instability and dynamic response of MEMS [11]. Sedighi [12] proposed a new model to investigate the size-dependent dynamic pull-in instability of vibrating electrically actuated micro-beams based on the strain gradient elasticity theory. Next, Ansari et al. [13] have investigated the size-dependent pull-in response of hydrostatically and electrostatically actuated rectangular nano-plates including surface stress effects. Ramezani et al. [14] developed a closed-form solution of the pull-in instability in nano-cantilevers subjected to electrostatic and intermolecular surface forces. Fu and Zhang [15] studied size-dependent pull-in phenomenon in electrically actuated nano-beams. Wang et al. [8] investigated the extensional multi-layer micro-beams in the presence of pull-in phenomenon and vibrations. Sadeghian et al. [16] discussed application of electrostatic pull-in instability on sensing adsorbate stiffness in nano-
mechanical resonators. Shaat and Abdelkefi [17] have studied pull-in instability of multi-phase nanocrystalline silicon beams under distributed electrostatic force. Hasanyan et al. [18] determined pull-in instabilities in functionally graded (FG) microthermoelectro mechanical systems. Due to the countless usages of double-clamped micro-beams in MEMS, this type of beams is analyzed in this paper.

It is noted that present work investigates the pull-in instability of micro systems under the effects of input voltage, dynamic loading, and viscous damping. The Euler-Bernoulli beam theory and non-linear Reynolds equation are also employed to derive the non-linear partial differential governing equations. Furthermore, the generalized differential quadrature method (GDQM) is used to solve whole equations numerically. In conclusion, the results obtained from coupled equations are compared and effects of the pressure and applied voltage on the pull-in instability response of the actuated micro-beam have been reported.

2. FORMULATION OF THE PROBLEM

In this section we introduce the coupled partial differential equations that govern the behavior of the system. The structure under study is shown in Fig. 1; \( l \) denotes the length, \( b \) the width, \( h \) the thickness, and \( d \) the initial gap between micro-beam and fixed substrate.

According to the Fig. 1, the micro-beam is modeled within the Euler-Bernoulli beam theory. Following this theory, the governing equation of motion for deflection of micro-beam can be written as [19]

\[
EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + p(x,t)b = \frac{1}{2} \frac{\varepsilon b V^2}{(1-w(x,t))^2} + EA \left[ \int_0^l \left( \frac{\partial w(x,t)}{\partial x} \right)^2 dx + N_0 \right] \frac{\partial^2 w(x,t)}{\partial x^2} 
\]

where we have also assumed that the deflection is constant along the width. In the above equation, \( EI \) and \( EA \) both indicate the bending and axial stiffness, respectively; \( \rho A \) is mass per unit length, and \( \varepsilon, V, \) and \( N_0 \) are also dielectric constant of the gap medium, applied voltage, and axial load, respectively [19]. The micro-beam deflection is subjected to the following kinematic boundary conditions as

\[
w(0,t) = \frac{\partial w(0,t)}{\partial x} = 0
\]

\[
w(l,t) = \frac{\partial w(l,t)}{\partial x} = 0
\]

(2)

In MEMS, the space between the two plates (micro-beam and fixed substrate) is often filled with a fluid (e.g. air). To analyze the dynamics of the fluid between the micro-beam and fixed substrate, we assume that along the gap, the fluid always remains in contact with the micro-beam during the
motion [20]. It is apparent that the major mechanism of viscous damping for most MEMS/NEMS is squeeze-film damping (SQFD), which is governed by Reynolds equation as [21]

\[
\frac{\partial}{\partial x} \left( \frac{P h^3 \frac{\partial P}{\partial x}}{\mu} \right) + \frac{\partial}{\partial y} \left( \frac{P h^3 \frac{\partial P}{\partial y}}{\mu} \right) = 12 \frac{\partial (hP)}{\partial t}
\]

(3)

where \( \mu \) is the viscosity, \( h \) is the thickness of the film, and \( P \) is the pressure [21]. It should be noted that the pressure \( P \) consists of two parts, the ambient pressure \( P_a \) and a deviatory pressure \( p \) caused by the squeeze-film effect [22]. The corresponding boundary conditions at the clamped edge of the micro-beam state as

\[
\frac{\partial P(0,t)}{\partial x} = \frac{\partial P(l,t)}{\partial x} = 0
\]

(4)

For convenience, the following dimensionless variables are introduced

\[
W = \frac{w}{d_0}, \quad X = \frac{x}{l}, \quad T = \frac{t}{\tau}, \quad P = \frac{P}{P_a}, \quad H = \frac{d}{d_0}
\]

(5)

It should be noted that \( d_0 = d + w \). Substituting the dimensionless variables given in (5) into (1) and (3), the dimensionless governing equations are expressed as

\[
\frac{\partial^2 W(X,T)}{\partial X^2} + \beta P(X,T) = \alpha_2 \frac{V^2}{(1-W(X,T))^2} + \left[ \alpha_1 \int_0^1 \left( \frac{\partial W(X,T)}{\partial X} \right)^2 dX + N \right] \frac{\partial W(X,T)}{\partial X^2}
\]

(6)

\[
\frac{\partial^2 P(X,T)}{\partial X^2} = \sigma \left[ \frac{\partial P}{\partial T} + \frac{\partial H}{\partial T} \right]
\]

(7)

where

\[
\alpha_i = 6 \left( \frac{h}{d_0} \right)^2, \quad N = \frac{\rho l^4}{EI}, \quad \alpha_2 = 6 \frac{\rho l^4}{h^2 E d_0^2}, \quad \tau = \sqrt{\frac{\rho A l^4}{E A}}
\]

\[
\beta = \frac{P h^4}{E d_0}, \quad \sigma = \frac{12 \mu}{P d_0^2 \sqrt{\rho b h \frac{E l}{E l}}}
\]

(8)

In addition, the corresponding boundary conditions can be written in the following dimensionless variables

\[
W(0,T) = W(l,T) = \frac{\partial W(0,T)}{\partial X} = \frac{\partial W(l,T)}{\partial X} = 0
\]

(9)

\[
\frac{\partial P(0,T)}{\partial X} = \frac{\partial P(l,T)}{\partial X} = 0
\]

(10)

3. NUMERICAL SOLUTION

The GDQM consists of the approximation of partial derivatives of a function at a point is affected by the values of the function in the total domain. According to GDQM, the first derivative of the solution function can be determined by [23].
Shu and Chew [24] showed that one of the best options for obtaining grid points is zeros of the well-known Chebyshev polynomials as

$$x_i = \frac{1}{2} \left(1 - \cos\left(\frac{i - 1}{N - 1}\pi\right)\right) \quad \text{for} \quad i = 1, 2, ..., N$$

In addition, the weighting coefficients of higher order derivatives can also be expressed as [23].

$$L^{(n)}_i = n \left(L^{(n-1)}_{ij} - \frac{L^{(n-1)}_{ij}}{x_i - x_j}\right) \quad \text{for} \quad \left\{\begin{array}{l} i, j = 1, 2, ..., N \\ n = 2, 3, ..., N - 1 \\ j \neq i \end{array}\right.$$  

$$L^{(n)}_i = \sum_{j \neq i} L^{(n)}_{ij} \quad \text{for} \quad \left\{\begin{array}{l} i = 1, 2, ..., N \\ n = 1, 2, ..., N - 1 \end{array}\right.$$  

For solving the problem, using integration by parts we have

$$\int_0^1 \left(\frac{\partial W}{\partial X} (X, T)\right)^2 dX = -\int_0^1 \frac{\partial^2 W}{\partial X^2} (X, T) W (X, T) dX$$  

By applying Newton-Cotes formula [24] on (14), we can define

$$\int_0^1 \frac{\partial^2 W}{\partial X^2} (X, T) W (X, T) dX \approx \sum_{i=1}^M A_i W_i$$  

where

$$A_i = \int_0^1 \prod_{j \neq i} \frac{X - X_j}{X_i - X_j} dX$$  

Therefore, using GDQ method, the governing equations can be obtained as

$$\sum_{j=1}^N L^{(4)}_{ij} W_j + \dot{W}_i = \left(N - \alpha_i \sum_{m=1}^N \sum_{n=1}^N A_{mn} W_m L^{(2)}_{ij} W_n\right)$$

$$\times \sum_{j=1}^N L^{(2)}_{ij} W_j + \alpha_2 \frac{V^2}{(1-W_i)^2} - \beta P$$

$$H^3 P \sum_{j=1}^N L^{(2)}_{ij} P_j + 3H^2 P \sum_{j=1}^N L_{ij} W_j$$

$$= 12 \mu \left(P \dot{W}_i + H \dot{P}_i\right)$$
\[ W(X,T)|_{x=0} - W(X,T)|_{x=L} = 0 \]
\[ \sum_{j=1}^{N} L_j W_j = \sum_{j=1}^{N} L_j y_j = 0 \]
\[ \sum_{j=1}^{N} L_j P_j = \sum_{j=1}^{N} L_j y_j = 0 \]  

(19)

4. RESULTS AND DISCUSSION

In this section, we indicate the methodology used for the numerical solution of the governing equations. It is noted that the system parameters for the pull-in analysis of the poly-silicon micro-beam are reported in Table 1. To demonstrate the effect of input voltage, the Newton-Raphson iterative algorithm is used to solve the non-linear algebraic equations system. Pull-in input voltage for this micro-beam without the effect of applied load is considered as \( V_{PI} = 9 \text{ V} \) which agrees very well with static pull-in voltage [25]. In other words, dynamic loading makes micro-beam pulled into the fixed electrode at higher values of input voltage. However, for simplicity, Newmark’s integration scheme [26] is used in current paper as it solves a dynamic problem in different time steps.

**TABLE 1. GEOMETRY AND MATERIAL PROPERTIES OF POLY-SILICON MICRO-BEAM.**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l )</td>
<td>600</td>
<td>( \mu \text{m} )</td>
</tr>
<tr>
<td>( b )</td>
<td>40</td>
<td>( \mu \text{m} )</td>
</tr>
<tr>
<td>( h )</td>
<td>2.3</td>
<td>( \mu \text{m} )</td>
</tr>
<tr>
<td>( d )</td>
<td>2.1</td>
<td>( \mu \text{m} )</td>
</tr>
<tr>
<td>( E )</td>
<td>166</td>
<td>GPa</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( 1.82 \times 10^{-7} )</td>
<td>( \text{N} / (\text{m} \cdot \text{m}) )</td>
</tr>
<tr>
<td>( v )</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>( 8.854 \times 10^{-12} )</td>
<td>( \text{C}^2 / (\text{N} \cdot \text{m}) )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>2330</td>
<td>( \text{kg} / (\text{m} \cdot \text{m}) )</td>
</tr>
</tbody>
</table>

Fig. 2 exhibits the comparison between linear and non-linear analysis of dynamic pull-in instability response of the poly-silicon micro-beam in vacuum pressure. It is noted that pressure distribution is considered constant through the initial gap. The pull-in time response decreases when the applied voltage reaches a certain value. It is worth nothing that the certain value of voltage leads to pull-in voltage. Moreover, it is apparent that both curves indicate decreasing of pull-in time response. It is due to the fact that by reduction of the strength of fluid, poly-silicon micro-beam can be actuated much more conveniently. In addition, it should be noted that higher order terms of von-Kármán non-linearity strains have influence on the final response of the oscillatory system.

**Fig. 2.** Comparison between linear and non-linear analysis for a micro-beam with properties listed in Table 1.
Plotted in Fig. 3 is the difference between pull-in time of micro-beam versus applied voltage for various ambient pressures. It can be seen that by increasing the value of pressure, pull-in time response will be increased. In addition, it is demonstrated that in all cases, after applying critical voltage on the system, final response will be stable.

\[
\begin{align*}
\frac{\partial^2 W(X,T)}{\partial X^2} + \rho A \frac{\partial^2 W(X,T)}{\partial T^2} + \beta P(X,T) \\
= \left[ \alpha \int \left( \frac{\partial W(X,T)}{\partial X} \right)^2 dX + N \right] \frac{\partial W(X,T)}{\partial X^2}
\end{align*}
\]  

(20)

It can be concluded that existing small difference between these two curves is due to the linear and non-linear Reynolds equation.

Another comparison is also performed here to show the effects of viscous damping when the electrostatic force has been neglected. To this end, the equation of motion for micro-beam is considered as

Fig. 4. Pull-in response of the system without electrostatic actuation with respect to Reynolds equation.
Veijola [27] proposed the relationship between viscosity and Knudsen number as

$$\mu_{eff} = \frac{\mu}{1 + 9.638K^n}$$

(21)

To investigate the effects of viscosity on the stable response of the system, variation of pull-in time versus ambient pressure are illustrated in Fig. 5. According to this figure and (21), by increasing the value of ambient pressure, the pull-in time is increased. Furthermore, it is shown that considering the fluid as a continuum is not true because in this pressure domain, energy will be decreased.

**Conclusions**

In the present work, the pull-in instability response of poly-silicon micro-beam subjected to electrostatic actuation and SQFD was studied. The corresponding governing equations and boundary conditions are derived by using the Euler-Bernoulli beam theory and non-linear Reynolds equation. Afterwards, the GDQM is implemented to discretize the governing differential equations along various boundary conditions. In addition, Newmark’s integration scheme and Newton-Raphson iterative algorithm were employed. It was found that when the applied voltage reaches the specific value, the divergence instability occurs. It should also be noted that neglecting of pressure may lead to a smaller values of pull-in time. Furthermore, it was demonstrated that in all cases, the pull-in time response for this micro-beam is size-dependent. The investigation will be useful in designing microsystems on pull-in instability under the effects of viscous damping, input voltage, and electrostatic force.

**REFERENCES**


